Maximal Independent Sets in Clique-free Graphs

Sam Spiro, UC San Diego.

Joint with Xiaoyu He and Jiaxi Nie

History

Given a graph G, we say that a set $I \subseteq V(G)$ is a *maximal independent set* (abbreviated MIS) if I is an independent set but $I \cup \{v\}$ is not for any $v \notin I$.

History

Given a graph G, we say that a set $I \subseteq V(G)$ is a *maximal independent set* (abbreviated MIS) if I is an independent set but $I \cup \{v\}$ is not for any $v \notin I$.

There has been a lot of work on algorithmic aspects of MIS's, in part due to application in social network analysis, bioinformatics, information retrieval, computer vision,...

History

Given a graph G, we say that a set $I \subseteq V(G)$ is a *maximal independent set* (abbreviated MIS) if I is an independent set but $I \cup \{v\}$ is not for any $v \notin I$.

There has been a lot of work on algorithmic aspects of MIS's, in part due to application in social network analysis, bioinformatics, information retrieval, computer vision,...

Question

Given a family of graphs $\mathcal{G}$, what's the maximum number of MIS's that a graph $G \in \mathcal{G}$ can have?

History

Let $m(n)$ denote the maximum number of MIS's in an n-vertex graph.

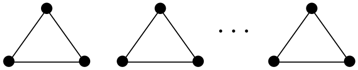
History

Let $m(n)$ denote the maximum number of MIS's in an n-vertex graph.

Theorem (Miller, Muller 1960; Moon, Moser 1965)

If $n \geq 2$, then

$$m(n) = \begin{cases} 3^{n/3} & n \equiv 0 \mod 3, \\ 4 \cdot 3^{(n-4)/3} & n \equiv 1 \mod 3, \\ 2 \cdot 3^{(n-2)/3} & n \equiv 2 \mod 3. \end{cases}$$



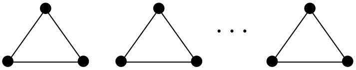
History

Let $m(n)$ denote the maximum number of MIS's in an n-vertex graph.

Theorem (Miller, Muller 1960; Moon, Moser 1965)

If $n \geq 2$, then

$$m(n) = \begin{cases} 3^{n/3} & n \equiv 0 \mod 3, \\ 4 \cdot 3^{(n-4)/3} & n \equiv 1 \mod 3, \\ 2 \cdot 3^{(n-2)/3} & n \equiv 2 \mod 3. \end{cases}$$



What happens if we consider graphs which are "far" from this extremal construction?

History

Let $m_3(n)$ denote the maximum number of MIS's in an n-vertex triangle-free graph.

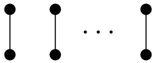
History

Let $m_3(n)$ denote the maximum number of MIS's in an n-vertex triangle-free graph.

Theorem (Hujter, Tuza 1993)

If $n \geq 4$, then

$$m_3(n) = \begin{cases} 2^{n/2} & n \equiv 0 \mod 2, \\ 5 \cdot 2^{(n-5)/2} & n \equiv 1 \mod 2. \end{cases}$$



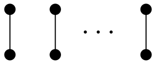
History

Let $m_3(n)$ denote the maximum number of MIS's in an n-vertex triangle-free graph.

Theorem (Hujter, Tuza 1993)

If $n \geq 4$, then

$$m_3(n) = \begin{cases} 2^{n/2} & n \equiv 0 \mod 2, \\ 5 \cdot 2^{(n-5)/2} & n \equiv 1 \mod 2. \end{cases}$$



This theorem (and variants thereof) have found applications in counting the number of maximal triangle-free graphs on n-vertices (Balogh-Petříčková) as well as to counting the number of maximal sum-free subsets (Balogh-Liu-Sharifzadeh-Treglown).

History

A somewhat more "refined" problem one can consider is counting the number of MIS's of a given size k, which we will refer to as k-MIS's.

History

A somewhat more “refined” problem one can consider is counting the number of MIS’s of a given size k, which we will refer to as k-MIS’s. Let $m(n, k)$ denote the maximum number of k-MIS’s that an n-vertex graph can have.

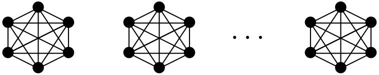
History

A somewhat more "refined" problem one can consider is counting the number of MIS's of a given size k, which we will refer to as k-MIS's. Let $m(n,k)$ denote the maximum number of k-MIS's that an n-vertex graph can have.

Theorem (Nielsen 2002)

If $s \in \{0, 1, \ldots, k-1\}$ *with* $n \equiv s \mod k$, *then*

$$m(n,k) = \lfloor n/k \rfloor^{k-s} \lceil n/k \rceil^{s}.$$



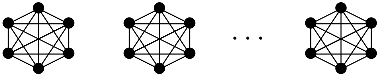
History

A somewhat more "refined" problem one can consider is counting the number of MIS's of a given size k, which we will refer to as k-MIS's. Let $m(n,k)$ denote the maximum number of k-MIS's that an n-vertex graph can have.

Theorem (Nielsen 2002)

If $s \in \{0, 1, \ldots, k-1\}$ with $n \equiv s \mod k$, then

$$m(n,k) = \lfloor n/k \rfloor^{k-s} \lceil n/k \rceil^{s}.$$



Again one can ask how this function changes if we consider graphs which are "far" from the disjoint union of cliques.

Clique-free Graphs

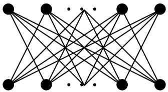
Define $m_t(n, k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have.

Clique-free Graphs

Define $m_t(n, k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have. Given the previous constructions, we might expect that the maximizer for $m_t(n, k)$ will consist of the disjoint union of some "nice" graphs.

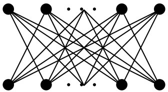
Clique-free Graphs

Define $m_t(n,k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have. Given the previous constructions, we might expect that the maximizer for $m_t(n,k)$ will consist of the disjoint union of some "nice" graphs. One natural construction that comes up is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ minus a maximum matching, which we will refer to as a *comatching*.



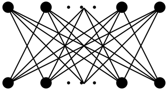
Clique-free Graphs

Define $m_t(n,k)$ to be the maximum number of k-MIS's that an n-vertex K_t-free graph can have. Given the previous constructions, we might expect that the maximizer for $m_t(n,k)$ will consist of the disjoint union of some "nice" graphs. One natural construction that comes up is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ minus a maximum matching, which we will refer to as a *comatching*.



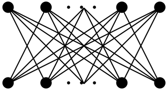
Note that the comatching has (at least) $\lfloor n/2 \rfloor$ 2-MIS's.

Clique-free Graphs

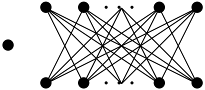


$$m_3(n, 2) = \Omega(n)$$

Clique-free Graphs

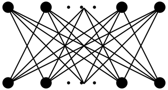


$$m_3(n,2) = \Omega(n)$$

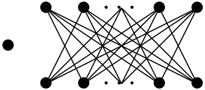


$$m_3(n,3) = \Omega(n)$$

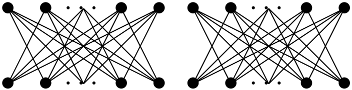
Clique-free Graphs



$$m_3(n,2) = \Omega(n)$$

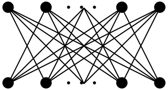


$$m_3(n,3) = \Omega(n)$$

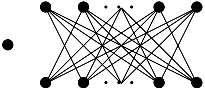


$$m_3(n,4) = \Omega(n^2)$$

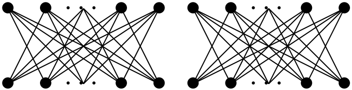
Clique-free Graphs



$$m_3(n, 2) = \Omega(n)$$



$$m_3(n, 3) = \Omega(n)$$



$$m_3(n, 4) = \Omega(n^2)$$

More generally this shows $m_t(n, k) = \Omega(n^{\lfloor k/2 \rfloor})$ for fixed k.

Clique-free Graphs

Reasonable Question

Is it the case that for all k, t we have

$$m_t(n, k) = O_{k,t}(n^{\lfloor k/2 \rfloor}).$$

Clique-free Graphs

Reasonable Question

Is it the case that for all k, t we have

$$m_t(n, k) = O_{k,t}(n^{\lfloor k/2 \rfloor}).$$

Theorem (He, Nie, S. 2021)

For $n \geq 8$ we have

$$m_3(n, 2) = \lfloor n/2 \rfloor,$$

and the unique graph achieving this bound is a comatching of order n.

Clique-free Graphs

Reasonable Question

Is it the case that for all k, t we have

$$m_t(n,k) = O_{k,t}(n^{\lfloor k/2 \rfloor}).$$

Theorem (He, Nie, S. 2021)

For $n \geq 8$ we have

$$m_3(n,2) = \lfloor n/2 \rfloor,$$

and the unique graph achieving this bound is a comatching of order n. Moreover, we have

$$m_3(n,3) = \Theta(n),$$
$$m_3(n,4) = \Theta(n^2).$$

Better Constructions

Proposition

For all $t \geq 4$,

$$m_t(n, 3) \geq n^{2-o(1)}.$$

Better Constructions

Proposition

For all $t \geq 4$,

$$m_t(n, 3) \geq n^{2-o(1)}.$$

A famous result of Ruzsa and Szemerédi says that there exists an n-vertex tripartite graph G on $U \cup V \cup W$ with $n^{2-o(1)}$ edges such that every edge is contained in a unique triangle.

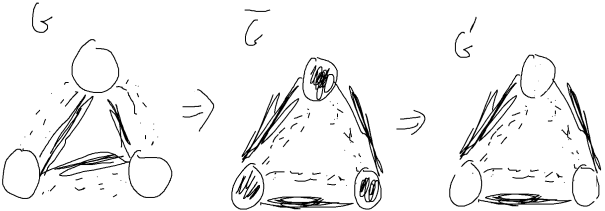
Better Constructions

Proposition

For all $t \geq 4$,

$$m_t(n,3) \geq n^{2-o(1)}.$$

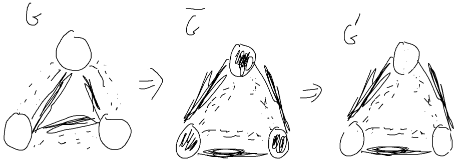
A famous result of Ruzsa and Szemerédi says that there exists an n-vertex tripartite graph G on $U \cup V \cup W$ with $n^{2-o(1)}$ edges such that every edge is contained in a unique triangle. Let G' be the "tripartite complement" of G, i.e. take the complement $\bar{G}$ and then delete all the edges within each of the parts U, V, W.



Better Constructions

Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'.

Better Constructions



Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'. It's an independent set since it's the (tripartite) complement of a triangle.

Better Constructions

Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'. It's an independent set since it's the (tripartite) complement of a triangle. If $\{u, v, w, w'\}$ is an independent set in G', then $\{u, v, w'\}$ is a triangle in G.

Better Constructions

Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'. It's an independent set since it's the (tripartite) complement of a triangle. If $\{u, v, w, w'\}$ is an independent set in G', then $\{u, v, w'\}$ is a triangle in G. Because there is a unique triangle in G containing uv, we must have $w' = w$, i.e. this is maximal.

Better Constructions

Claim: every triangle $T = \{u, v, w\}$ in G is a 3-MIS in G'. It's an independent set since it's the (tripartite) complement of a triangle. If $\{u, v, w, w'\}$ is an independent set in G', then $\{u, v, w'\}$ is a triangle in G. Because there is a unique triangle in G containing uv, we must have $w' = w$, i.e. this is maximal. Since G contains $n^{2-o(1)}$ triangles, and since the tripartite graph G' is K_t-free for $t \geq 4$, we conclude the result. □

Better Constructions

Mimicking this proof strategy, we can improve our bounds if there exists k-partite n-vertex graphs with many copies of K_{k-1} which are all contained in a unique K_k.

Better Constructions

Mimicking this proof strategy, we can improve our bounds if there exists k-partite n-vertex graphs with many copies of K_{k-1} which are all contained in a unique K_k.

Theorem (Gowers, B. Janzer 2020)

For all $1 \leq \ell < k$ there exist n-vertex graphs with $n^{\ell - o(1)}$ copies of K_ℓ such that every K_ℓ is contained in at most one K_k.

Better Constructions

Mimicking this proof strategy, we can improve our bounds if there exists k-partite n-vertex graphs with many copies of K_{k-1} which are all contained in a unique K_k.

Theorem (Gowers, B. Janzer 2020)

For all $1 \leq \ell < k$ there exist n-vertex graphs with $n^{\ell - o(1)}$ copies of K_ℓ such that every K_ℓ is contained in at most one K_k.

Taking $\ell = k - 1$ gives the following:

Proposition

For $k < t$ we have

$$m_t(n, k) \geq n^{k-1-o(1)}.$$

Better Constructions

By taking disjoint unions of these constructions (like we did with K_1 and comatchings) gives the following:

Theorem (He, Nie, S. 2021)

For all fixed k, t, we have

$$m_t(n,k) \geq n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor - o(1)}.$$

Better Constructions

By taking disjoint unions of these constructions (like we did with K_1 and comatchings) gives the following:

Theorem (He, Nie, S. 2021)

For all fixed k, t, we have

$$m_t(n,k) \geq n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor - o(1)}.$$

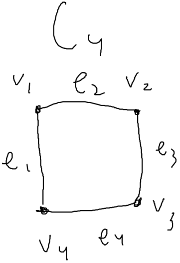
Reasonable Question

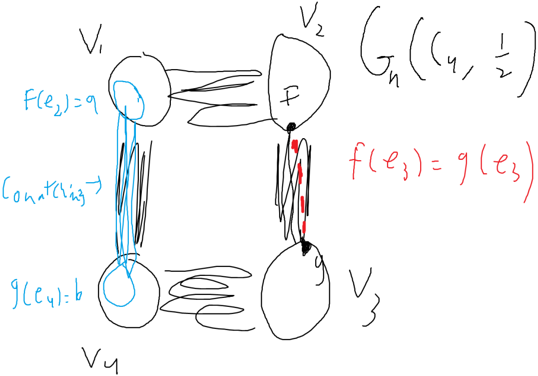
Is this bound essentially tight? In particular, for triangle-free graphs do we have

$$m_3(n,k) = \Theta(n^{\lfloor k/2 \rfloor}).$$

Better Construction: Blowups

Consider a C_k with edges $e_1, \ldots, e_k$ and vertices $v_i \in e_i \cap e_{i+1}$ and define $G_n(C_k, \frac{1}{2})$ as follows. Let V_i consist of the set of functions $f : \{e_i, e_{i+1}\} \to [n^{1/2}]$. We make $f \in V_i$ adjacent to $g \in V_j$ if and only if there is an edge $e \in E(C_k)$ with $v_i, v_j \in e$ (i.e. if $i = j \pm 1$) and such that $f(e) \neq g(e)$.

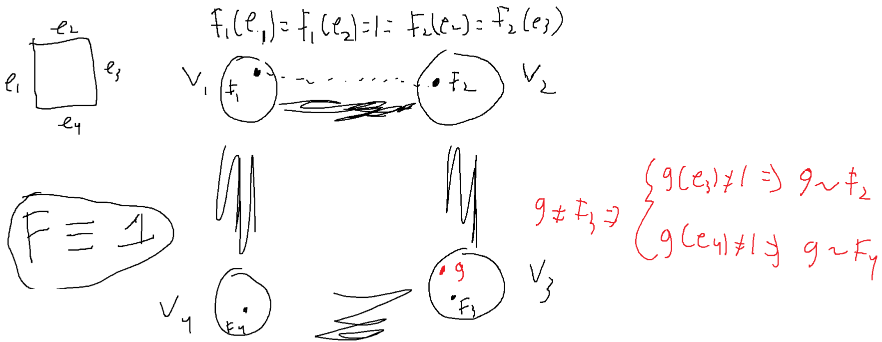




Better Construction: Blowups

Lemma

For each function $F : E(C_k) \to [n^{1/2}]$, the set I of $f \in V(G_n(C_k, \frac{1}{2}))$ which agree with F forms a k-MIS.



Better Construction: Blowups

These blowups have kn vertices, have at least $n^{k/2}$ k-MIS's (i.e. functions $F : E(C_k) \to [n^{1/2}]$), and they are triangle-free for $k > 3$.

Better Construction: Blowups

These blowups have kn vertices, have at least $n^{k/2}$ k-MIS's (i.e. functions $F : E(C_k) \to [n^{1/2}]$), and they are triangle-free for $k > 3$.

Theorem (He, Nie, S. 2021)

For all $k \geq 4$,

$$m_3(n,k) = \Omega(n^{k/2}).$$

Better Construction: Blowups

These blowups have kn vertices, have at least $n^{k/2}$ k-MIS's (i.e. functions $F : E(C_k) \to [n^{1/2}]$), and they are triangle-free for $k > 3$.

Theorem (He, Nie, S. 2021)

For all $k \geq 4$,

$$m_3(n, k) = \Omega(n^{k/2}).$$

One can generalize this blowup construction in two ways.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H, M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H, M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

For example, the previous construction used $H = C_k$ and $M \equiv \frac{1}{2}$.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H, M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

For example, the previous construction used $H = C_k$ and $M \equiv \frac{1}{2}$. For $H = C_{2k}$, taking M to be a perfect matching gives a disjoint union of comatchings.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H, M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

For example, the previous construction used $H = C_k$ and $M \equiv \frac{1}{2}$. For $H = C_{2k}$, taking M to be a perfect matching gives a disjoint union of comatchings. One can further interpolate between these two constructions by varying M.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H,M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

For example, the previous construction used $H = C_k$ and $M \equiv \frac{1}{2}$. For $H = C_{2k}$, taking M to be a perfect matching gives a disjoint union of comatchings. One can further interpolate between these two constructions by varying M.

If H is a k-vertex triangle-free graph and M has size $k/2$, then $G_n(H,M)$ will have kn vertices, $\Omega(n^{k/2})$ distinct k-MIS's, and be triangle-free.

Better Construction: Blowups

Given any k-vertex graph H and a fractional matching M, define $G_n(H, M)$ to be the graph on $\bigcup_{u \in V(H)} V_u$ where V_u are the set of functions f which map $e \ni u$ to $[n^{M(e)}]$, and we make $f \in V_u, g \in V_w$ adjacent iff $uw \in E(H)$ and $f(uw) \neq g(uw)$.

For example, the previous construction used $H = C_k$ and $M \equiv \frac{1}{2}$. For $H = C_{2k}$, taking M to be a perfect matching gives a disjoint union of comatchings. One can further interpolate between these two constructions by varying M.

If H is a k-vertex triangle-free graph and M has size $k/2$, then $G_n(H, M)$ will have kn vertices, $\Omega(n^{k/2})$ distinct k-MIS's, and be triangle-free. Thus if this bound is tight, there are many different constructions achieving it.

Better Construction: Blowups

One can also generalize this construction by taking “blowups” of hypergraphs H.

Better Construction: Blowups

One can also generalize this construction by taking “blowups” of hypergraphs H. Essentially in this setting one replaces hyperedges of H by “interwoven” copies of the Ruzsa-Szemerédi and Gowers-Janzer constructions instead of “interwoven” comatchings.

Better Construction: Blowups

One can also generalize this construction by taking "blowups" of hypergraphs H. Essentially in this setting one replaces hyperedges of H by "interwoven" copies of the Ruzsa-Szemerédi and Gowers-Janzer constructions instead of "interwoven" comatchings. With this construction and H the tight cycle, one can prove the following:

Theorem (He, Nie, S. 2021)

$t \geq 3$ and $k \geq 2(t-1)$, then

$$m_t(n,k) \geq n^{\frac{(t-2)k}{t-1} - o(1)}.$$

Note that this drops the floor from the previous bound.

Upper Bounds

We think these lower bounds are essentially best possible:

Conjecture (He, Nie, S.; S.)

For all fixed k, t, we have

$$m_t(n,k) = O(n^{\frac{(t-2)k}{t-1}}).$$

Moreover, for $k < 2(t-1)$ we have

$$m_t(n,k) = O(n^{\lfloor \frac{(t-2)k}{t-1} \rfloor}).$$

Upper Bounds

Proposition

For all $k < t$ we have

$$m_t(n,k) = O(n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor}) = O(n^{k-1}).$$

Upper Bounds

Proposition

For all $k < t$ we have

$$m_t(n,k) = O(n^{\left\lfloor \frac{(t-2)k}{t-1} \right\rfloor}) = O(n^{k-1}).$$

It's easy to prove $m_t(n,1) \leq t$, and then one can inductively use that (roughly) $m_t(n,k) \leq n \cdot m_t(n,k-1)$.

Upper Bounds

Theorem (He, Nie, S. 2021)

For all $k \leq 4$*, we have*

$$m_3(n,k) = O(n^{\lfloor k/2 \rfloor}).$$

Upper Bounds

Theorem (He, Nie, S. 2021)

For all $k \leq 4$, we have

$$m_3(n,k) = O(n^{\lfloor k/2 \rfloor}).$$

The case $k \leq 2$ follows from the previous proposition, and the $k = 4$ case will follow from the $k = 3$ case since (roughly) $m_3(n,4) \leq n \cdot m_3(n,3)$. Thus it remains to prove this for $k = 3$.

Upper Bounds

If G is a k-partite graph, let $t(G, k)$ denote the number of "transversal MIS's", i.e. MIS's using exactly one vertex from each of the k parts of G.

Upper Bounds

If G is a k-partite graph, let $t(G,k)$ denote the number of "transversal MIS's", i.e. MIS's using exactly one vertex from each of the k parts of G.

Lemma

If G is a triangle-free graph on n vertices, then G has an induced k-partite subgraph $G' \subseteq G$ satisfying

$$t(G',k) \geq (4k)^{-k} m(G,k),$$

where $m(H,k)$ denotes the number of k-MIS's of H.

Upper Bounds

If G is a k-partite graph, let $t(G, k)$ denote the number of "transversal MIS's", i.e. MIS's using exactly one vertex from each of the k parts of G.

Lemma

If G is a triangle-free graph on n vertices, then G has an induced k-partite subgraph $G' \subseteq G$ satisfying

$$t(G', k) \geq (4k)^{-k} m(G, k),$$

where $m(H, k)$ denotes the number of k-MIS's of H.

That is, to determine $m_3(n, k)$ it suffices to restrict our attention to counting transversal MIS's in k-partite triangle-free graphs.

Upper Bounds

Lemma

If G is an n-vertex triangle-free 3-partite graph on $U \cup V \cup W$, then it contains at most n transversal MIS's.

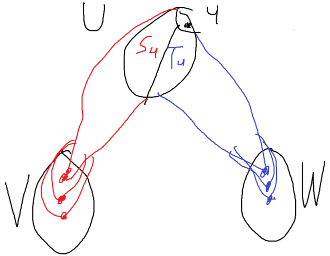
Upper Bounds

Lemma

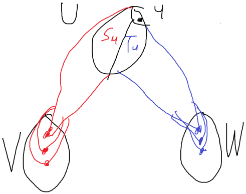
If G is an n-vertex triangle-free 3-partite graph on $U \cup V \cup W$, then it contains at most n transversal MIS's.

Claim

For every $u \in U$ in at least one transversal MIS, there exist unique sets $S_u, T_u \subseteq U$ such that any MIS $(u, v, w) \in U \times V \times W$ satisfies $N(v) \cap U = S_u$ and $N(w) \cap U = T_u$.

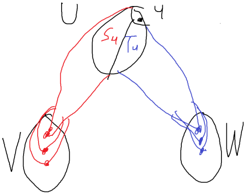


Upper Bounds



Define $V(S) = \{v \in V : N(v) \cap U = S\}$ and similarly $W(T)$. Let $U_{\geq 2} \subseteq U$ be the vertices in at least two transversal MIS's.

Upper Bounds



Define $V(S) = \{v \in V : N(v) \cap U = S\}$ and similarly $W(T)$. Let $U_{\geq 2} \subseteq U$ be the vertices in at least two transversal MIS's.

Claim

The number of transversal MIS's using $u \in U_{\geq 2}$ is at most

$$\sum_{u \in U_{\geq 2}} \min\{|V(S_u)|, |W(T_u)|\}.$$

Upper Bounds

Claim

In the sum

$$\sum_{u \in U_{\geq 2}} \min\{|V(S_u)|, |W(T_u)|\},$$

it is possible to use the upper bounds

$$\min\{|V(S_u)|, |W(T_u)|\} \leq |V(S_u)|, |W(T_u)|$$

such that each term appears at most once in the sum.

Upper Bounds

Claim

In the sum

$$\sum_{u \in U_{\geq 2}} \min\{|V(S_u)|, |W(T_u)|\},$$

it is possible to use the upper bounds

$$\min\{|V(S_u)|, |W(T_u)|\} \leq |V(S_u)|, |W(T_u)|$$

such that each term appears at most once in the sum.

This implies that the number of MIS's using vertices of $U_{\geq 2}$ are at most

$$\sum_{S} |V(S)| + \sum_{T} |W(T)| = |V| + |W|.$$

Upper Bounds

Claim

In the sum

$$\sum_{u \in U_{\geq 2}} \min\{|V(S_u)|, |W(T_u)|\},$$

it is possible to use the upper bounds

$$\min\{|V(S_u)|, |W(T_u)|\} \leq |V(S_u)|, |W(T_u)|$$

such that each term appears at most once in the sum.

This implies that the number of MIS's using vertices of $U_{\geq 2}$ are at most

$$\sum_S |V(S)| + \sum_T |W(T)| = |V| + |W|.$$

This doesn't count MIS's which have $u \notin U_{\geq 2}$, giving an extra count of at most $|U|$.

Open Problems: Order of Magnitude

There are many, many open problems left to explore.

Open Problems: Order of Magnitude

There are many, many open problems left to explore.

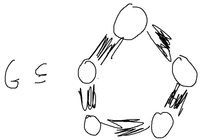
Conjecture

$$m_3(n, 5) = \Theta(n^{5/2}).$$

Open Problems: Order of Magnitude

Proposition (He, Nie, S. 2021)

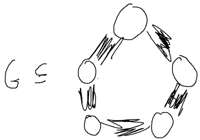
If G is an n-vertex graph which is the subgraph of a blowup of C_5, then it contains at most $O(n^{5/2})$ 5-MIS's.



Open Problems: Order of Magnitude

Proposition (He, Nie, S. 2021)

If G is an n-vertex graph which is the subgraph of a blowup of C_5, then it contains at most $O(n^{5/2})$ 5-MIS's.

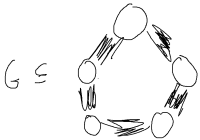


In particular, one can not improve our lower bound of $\Omega(n^{5/2})$ by coming up with a "better" blowup-like construction for C_5.

Open Problems: Order of Magnitude

Proposition (He, Nie, S. 2021)

If G is an n-vertex graph which is the subgraph of a blowup of C_5, then it contains at most $O(n^{5/2})$ 5-MIS's.



In particular, one can not improve our lower bound of $\Omega(n^{5/2})$ by coming up with a "better" blowup-like construction for C_5.

Conjecture

If G is an n-vertex subgraph of a blowup of a k-vertex triangle-free graph H, then G contains at most $O(n^{k/2})$ k-MIS's.

Open Problems: Order of Magnitude

Conjecture

There exists a $k > 4$ *such that*

$$m_3(n, k) = O(n^{k-3}).$$

Open Problems: Order of Magnitude

Conjecture

There exists a $k > 4$ such that

$$m_3(n, k) = O(n^{k-3}).$$

Our current best (non-trivial) bound is $O(n^{k-2})$, and we conjecture that the real answer is $O(n^{k/2})$.

Open Problems: Order of Magnitude

Question

For any fixed t, is it true that

$$m_t(n,t) = O(n^{t-2}).$$

Open Problems: Order of Magnitude

Question

For any fixed t, is it true that

$$m_t(n, t) = O(n^{t-2}).$$

If true, one can easily extend this to show that our lower bounds are tight for $k < 2(t-1)$, i.e. when we only have a floor in the exponent.

Open Problems: Order of Magnitude

Question

For any fixed t, is it true that

$$m_t(n, t) = O(n^{t-2}).$$

If true, one can easily extend this to show that our lower bounds are tight for $k < 2(t-1)$, i.e. when we only have a floor in the exponent. We know this is true for $t = 3$, and it may be possible to extend our ideas to t-partite K_t-free graphs.

Open Problems: Order of Magnitude

Question

Are the $o(1)$ terms in our exponents necessary when $t \geq 4$? In particular, is it true that

$$m_4(n, 3) = n^{2-o(1)}.$$

Open Problems: Order of Magnitude

Question

Are the $o(1)$ terms in our exponents necessary when $t \geq 4$? In particular, is it true that

$$m_4(n, 3) = n^{2-o(1)}.$$

Proposition

If G is an n-vertex tripartite graph, then G has at most $n^{2-o(1)}$ 3-MIS's.

Open Problems: Order of Magnitude

Question

Are the $o(1)$ terms in our exponents necessary when $t \geq 4$? In particular, is it true that

$$m_4(n, 3) = n^{2-o(1)}.$$

Proposition

If G is an n-vertex tripartite graph, then G has at most $n^{2-o(1)}$ 3-MIS's.

This follows by reducing the problem to the Ruzsa-Szemerédi problem like before.

Open Problems: Order of Magnitude

Question

Are the $o(1)$ terms in our exponents necessary when $t \geq 4$? In particular, is it true that

$$m_4(n, 3) = n^{2-o(1)}.$$

Proposition

If G is an n-vertex tripartite graph, then G has at most $n^{2-o(1)}$ 3-MIS's.

This follows by reducing the problem to the Ruzsa-Szemerédi problem like before. It's possible that this same $o(1)$ term is necessary for K_4-free graphs in general, showing that $m_t(n, k)$ is intimately connected to the Ruzsa-Szemerédi problem.

Open Problems: Order of Magnitude

Question

If G is an n-vertex K_4-free graph with at least $n^{\lfloor 2k/3 \rfloor - \epsilon}$ k-MIS's, is it true that G has chromatic number $O_k(1)$?

Open Problems: Order of Magnitude

Question

If G is an n-vertex K_4-free graph with at least $n^{\lfloor 2k/3 \rfloor - \epsilon}$ k-MIS's, is it true that G has chromatic number $O_k(1)$?

If this, then one can reduce computing $m_4(n,k)$ to the k-partite setting.

Open Problems: Order of Magnitude

Question

If G is an n-vertex K_4-free graph with at least $n^{\lfloor 2k/3 \rfloor - \epsilon}$ k-MIS's, is it true that G has chromatic number $O_k(1)$?

If this, then one can reduce computing $m_4(n, k)$ to the k-partite setting. In particular this would give a positive answer to the previous problem, and it could probably be used to determine $m_4(n, 4)$ as well.

Open Problems: Order of Magnitude

Question

If G is an n-vertex K_4-free graph with at least $n^{\lfloor 2k/3 \rfloor - \epsilon}$ k-MIS's, is it true that G has chromatic number $O_k(1)$?

If this, then one can reduce computing $m_4(n, k)$ to the k-partite setting. In particular this would give a positive answer to the previous problem, and it could probably be used to determine $m_4(n, 4)$ as well.

Note that for K_3-free graphs it is easy to prove that if G has at least 1 k-MIS, then $\chi(G) \leq k + 1$

Open Problems: Asymptotics

In addition to order of magnitude, one could also ask for finer asymptotics values.

Question

Can one determine $m_t(n, 2)$ asymptotically for all t?

Open Problems: Asymptotics

In addition to order of magnitude, one could also ask for finer asymptotics values.

Question

Can one determine $m_t(n, 2)$ asymptotically for all t?

We can prove an upper bound of $\frac{1}{2}(t-1)n$ by a simple counting argument, and a lower bound of roughly $\frac{1}{4}(t-1)n$ by taking a comatching of size $2n/(t-1)$ and replacing each vertex with a clique of size $(t-1)/2$.

Open Problems: Asymptotics

Conjecture

$$m_3(n,3) \sim n.$$

Open Problems: Asymptotics

Conjecture

$$m_3(n, 3) \sim n.$$

There are several constructions giving these asymptotics: a comatching together with a K_2, a $K_{n/2,n/2}$ minus any 2-factor, and a "blowup" construction of P_2.

Open Problems: Large k

Question

Can one say anything about $m_3(n, k)$ when $k = cn$ for some constant c?

Open Problems: Large k

Question

Can one say anything about $m_3(n, k)$ when $k = cn$ for some constant c?

Proposition

If n is even and $2n/5 \leq k \leq n/2$, then

$$m_3(n, k) \geq (25/32)^{k-n/2} 2^{n/2}.$$

Open Problems: Large k

Question

Can one say anything about $m_3(n, k)$ when $k = cn$ for some constant c?

Proposition

If n is even and $2n/5 \leq k \leq n/2$, then

$$m_3(n, k) \geq (25/32)^{k-n/2} 2^{n/2}.$$

Construction: take $n - 2k$ copies of C_5 and $5k$ copies of K_2.

Open Problems: Large k

Question

Can one say anything about $m_3(n, k)$ when $k = cn$ for some constant c?

Proposition

If n is even and $2n/5 \leq k \leq n/2$, then

$$m_3(n, k) \geq (25/32)^{k-n/2} 2^{n/2}.$$

Construction: take $n - 2k$ copies of C_5 and $5k$ copies of K_2. This is motivated by the Hujter-Tuza triangle-free construction which consists of a perfect matching with a C_5 is n is odd, which suggests that C_5 and K_2 are the most "efficient" triangle-free components for finding MIS's.

Open Problems: Hypergraphs

Define $m_t^r(n,k)$ to be the maximum number of MIS's of size k in a K_t^r-free r-uniform hypergraph on n vertices.

Open Problems: Hypergraphs

Define $m_t^r(n,k)$ to be the maximum number of MIS's of size k in a K_t^r-free r-uniform hypergraph on n vertices. Somewhat surprisingly, we can completely determine the order of magnitude of $m_t^r(n,k)$ for $r \geq 3$.

Proposition (He, Nie, S. 2021)

For $n \geq 4$, we have

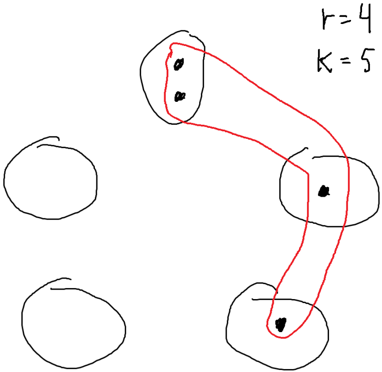
$$m_4^3(n,2) = n - 1.$$

For any other set of parameters satisfying $r \geq 3$, $k \geq r-1$, and $t \geq r+1$,

$$m_t^r(n,k) = \Theta_k(n^k).$$

Open Problems: Hypergraphs

Construction: split vertex set into blocks V_i of size n/k, take your hyperedges to be any set using two vertices from some V_i and one from each of $V_{i+1}, \ldots, V_{i+r-2}$.



Open Problems: Hypergraphs

Question

Can one determine sharper asymptotic bounds for $m_t^r(n,k)$?

Open Problems: Hypergraphs

Question

Can one determine sharper asymptotic bounds for $m_t^r(n, k)$?

Question

How many MIS's (in general or of a given size) can an n-vertex r-uniform hypergraph have?

Open Problems: Hypergraphs

Question

Can one determine sharper asymptotic bounds for $m_t^r(n, k)$?

Question

How many MIS's (in general or of a given size) can an n-vertex r-uniform hypergraph have?

The best construction I know of is to take a disjoint union of cliques of size $2r - 1$.

Open Problems: Hypergraphs

Question

Can one determine sharper asymptotic bounds for $m_t^r(n,k)$?

Question

How many MIS's (in general or of a given size) can an n-vertex r-uniform hypergraph have?

The best construction I know of is to take a disjoint union of cliques of size $2r-1$. Note that when $r=2$ this gives a disjoint union of triangles (which is tight).

The End

Thank You!